Lecture 4 - Objective and Outcomes

We continue our study of multivariate distributions by looking at conditioning, independence and joint moments.

- conditional distributions
- law of iterated expectations
- dependence and covariance

- sums of independent random variables
- joint moments

After reviewing the notes you should:

- be able to using conditioning to generate marginal distributions,
- know how to derive the moment generating function of a sum of independent random variables,
- understand how covariance fits into the framework of joint moments.

Discrete conditional distributions

1. Conditional distribution of Y given $X = x$:
   $F_{Y|X}(y|x) = P(Y \leq y|X = x)$.

2. Conditional mass of Y given $X = x$:
   $f_{Y|X}(y|x) = P(Y = y|X = x) = f_{X,Y}(x, y)/f_X(x)$.
3. Relationship between distribution and mass:
   $F_{Y|X}(y|x) = \sum_{y_i \leq y} f_{Y|X}(y_i|x)$.

Continuous conditional distributions

1. Conditional distribution of Y given $X = x$:
   $F_{Y|X}(y|x) = \int_{-\infty}^{y}(f_{X,Y}(x,v)/f_X(x))dv$.

2. Conditional density of Y given $X = x$:
   $f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x)$.

Conditional, joint and marginal densities

1. Conditional mass/density
   $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$
   $= \begin{cases} \frac{f_{X,Y}(x,y)}{\sum_y f_{X,Y}(x,y)}, & \text{discrete case,} \\ \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy}, & \text{continuous case.} \end{cases}$

2. Joint mass/density:
   $f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$.

3. Marginal mass/density

$$f_Y(y) = \begin{cases} \sum_x f_{Y|X}(y|x) f_X(x), & \text{discrete,} \\ \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx, & \text{continuous.} \end{cases}$$

4. Reverse conditioning

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_X(x)}{f_Y(y)} f_{Y|X}(y|x).$$

999 calls example

The probability that a 999 call arriving at a call center will be a hoax is p. The total number of calls arriving in any given day has a Poisson distribution with parameter λ. What is the distribution of the daily number of hoax calls?

Conditional expectation

Suppose X and Y random variables.

Condition expectation of Y given X: define

$$\psi(x) = E(Y|X = x)$$
$$= \begin{cases} \sum_y y f_{Y|X}(y|x), & \text{discrete,} \\ \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy, & \text{continuous.} \end{cases}$$

The conditional expectation of Y given X is $E(Y|X) = \psi(X)$ (a random variable).

Condition expectation of $g(Y)$ given X: If g is a well-behaved, real-valued function, define

$$h(x) = E(g(Y)|X = x)$$
$$= \begin{cases} \sum_y g(y) f_{Y|X}(y|x), & \text{discrete,} \\ \int_{-\infty}^{\infty} g(y) f_{Y|X}(y|x) dy, & \text{continuous.} \end{cases}$$

The conditional expectation of $g(Y)$ given X is $E(g(Y)|X) = h(X)$ (a random variable).

Law of iterated expectations

$$E[\psi(X)] = E[E(Y|X)] = E(Y).$$

Useful consequence,

$$E(Y) = \begin{cases} \sum_x E(Y|X=x) f_X(x), & \text{discrete,} \\ \int_{-\infty}^{\infty} E(Y|X=x) f_X(x) dx, & \text{continuous.} \end{cases}$$

Conditional variance

for random variables X and Y, define

$$\omega(x) = \mathrm{var}(Y|X=x)$$
$$= \begin{cases} \sum_y [y - E(Y|X=x)]^2 f_{Y|X}(y|x), & \text{discrete,} \\ \int_{-\infty}^{\infty} [y - E(Y|X=x)]^2 f_{Y|X}(y|x)dy, & \text{continuous.} \end{cases}$$

The conditional variance of Y given X is $\mathrm{var}(Y|X) = \omega(X) = E(Y^2|X) - [E(Y|X)]^2$. The conditional variance is a random variable and, using the law of iterated expectations, we can show that $E[\mathrm{var}(Y|X)] = \mathrm{var}(Y) - \mathrm{var}[E(Y|X)]$.

If $X_1, X_2, \ldots, X_n$ independent, then for all $x_1, x_2, \ldots, x_n$:

1. $F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = F_{X_1}(x_1)F_{X_2}(x_2)\ldots F_{X_n}(x_n)$,
2. $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\ldots f_{X_n}(x_n)$,
3. $E(X_1X_2\ldots X_n) = E(X_1)E(X_2)\ldots E(X_n)$,
4. $g_1(X_1), g_2(X_2), \ldots, g_n(X_n)$ are independent for real-valued functions $g_1, g_2, \ldots, g_n$.

Covariance function

for random variables X and Y,

$$\operatorname{cov}(X,Y) = E[(X - E(X))(Y - E(Y))]$$
$$= E(XY) - E(X)E(Y).$$

1. Symmetry:
   $\operatorname{cov}(X,Y) = \operatorname{cov}(Y,X)$.
2. With constant multipliers:
   $\operatorname{cov}(aX,bY) = ab\operatorname{cov}(X,Y)$.
3. Bilinearity:
   $\operatorname{cov}(X_1 + X_2, Y_1 + Y_2) =$

$\operatorname{cov}(X_1, Y_1) + \operatorname{cov}(X_1, Y_2) + \operatorname{cov}(X_2, Y_1) + \operatorname{cov}(X_2, Y_2)$.

4. Variance:

$$\operatorname{var}(X) = \operatorname{cov}(X, X),$$
$$\operatorname{var}(X + Y) = \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X, Y),$$
$$\operatorname{var}(X - Y) = \operatorname{var}(X) + \operatorname{var}(Y) - 2\operatorname{cov}(X, Y).$$

5. If X and Y are independent,

$$\operatorname{cov}(X, Y) = 0.$$

Sum of random variables

: for independent random variables X and Y, let $Z = X + Y$ be the sum. Density of sum is given by

$$f_Z(z) = \begin{cases} \sum_x f_X(x) f_Y(z-x), & \text{discrete,} \\ \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx, & \text{continuous.} \end{cases}$$

This type of function is known as a $convolution$.

Properties of joint moments

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1. r^{th} moment for X:
   $m_{r,0} = E(X^r)$.
2. r^{th} central moment for X:
   $\mu_{r,0} = E[(X - \mu_X)^r]$.
3. Covariance:
   $\mu_{1,1} = E[(X - E(X))(Y - E(Y))] = \mathrm{cov}(X, Y)$.